Euclidean tetrahedra and knot invariants

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Abstract

We construct knot invariants on the basis of ascribing Euclidean geometric values to a triangulation of sphere S^3 where the knot lies. The main new feature of this construction compared to the author's earlier papers on manifold invariants is that now nonzero "deficit angles" (in the terminology of Regge calculus) can also be handled. Moreover, the knot goes exactly along those edges of triangulations that have nonzero deficit angles.

1   Introduction

In this paper, we construct invariants of knots in the sphere S^3 based, very schematically, on the following ideas. A triangulation of S^3 is taken such that the knot goes along some of its edges. Then, Euclidean geometric values are ascribed in some way to the elements of triangulation. In particular, every edge gets a Euclidean lengths. Hence, the dihedral angles also acquire some values, as well as (say) the volumes of tetrahedra. These (dihedral angles and volumes) are taken with a minus sign for some tetrahedra, according to certain orientation considerations.

Then, an acyclic complex is constructed in the spirit of papers [8, 9], whose vector spaces consist of the differentials of our geometric values. Recall that in those papers invariants of three- and four-dimensional *manifolds* have been proposed based on algebraic relations corresponding in a natural way to simplicial (Pachner) moves — elementary rebuildings of a manifold triangulation. The principal new point in the present paper as compared with [8, 9] (as well as [3, 4, 7]) is that here an algebraic relation is proposed dealing with the situation where the *deficit angles* around some edges are nonzero modulo 2π. These deficit angles are defined like in Regge calculus: minus sum of the dihedral angles in all tetrahedra to which the edge belongs, save that now we take the *algebraic* sum, with the signs mentioned in the previous paragraph. Construction of a knot invariant follows as a natural application of such relation: the edges in the triangulation along which the knot goes are singled out exactly by the fact that the deficit angles for them are not zero but some value $(-\varphi)$.

Our knot invariant is expressed through the torsion of acyclic complex, edge lengths and tetrahedron volumes much like the three-manifold invariant [8, formula (5)] (edges belonging to the knot make, however, some difference, as we are going to explain).

Below, in section 2 we present simplicial moves which are enough to pass from a given triangulation of sphere S^3 *with a knot in it* to any other triangulation. In section 3, we present the relevant acyclic complex and the formula for knot invariant. In section 4, we prove our new key algebraic relation dealing with nonzero deficit angles. In the concluding section 5 we discuss our results and prospects for further work.

2   Simplicial moves

We are going to consider, strictly speaking, *pseudo*triangulations of a sphere with a knot. From triangulations in the proper sense of the word, they differ in that the boundary of a simplex in a pseudotriangulation can contain *several times* the same simplex of a smaller dimension; besides, a simplex in a pseudotriangulation is, generally, not determined by the list of its vertices. Note that although it is often more convenient for combinatorial topologists to consider triangulations in the proper sense (see especially the very useful paper [2]), it is usually not very hard to bridge the way to pseudotriangulations.

Let there be a knot K in the sphere S^3. Consider a pseudotriangulation of this sphere obeying the following conditions:

(a) the whole knot K lies on some edges of the pseudotriangulation;

(b) for any tetrahedron in the pseudotriangulation, not more than two of its vertices belong to K;

(c) any edge a of the pseudotriangulation either has two *different* vertices as its ends or, if its ends coincide, a represents a generator of the abelianization of the knot group $\pi(K)$.

In other words, condition (c) states that the edges with coinciding ends wind just one time around the knot. We use such edges to describe "moves $1 \leftrightarrow 2$" in the following theorem.

Theorem 1 *A pseudotriangulation of S^3 obeying the properties (a)–(c) can be transformed into any other pseudotriangulation with the same properties by a sequence of the following elementary moves:*

- *Pachner moves $2 \leftrightarrow 3$ and $1 \leftrightarrow 4$ (here, for instance, "$2 \leftrightarrow 3$" means that 2 tetrahedra are replaced with 3 tetrahedra or back). Such moves must not affect the edges lying on the knot K (the knot may, however, pass through edges and/or vertices lying in the boundary of the transformed cluster of tetrahedra);*

- *moves $1 \leftrightarrow 2$ "on the knot". Let an edge BD lie on the knot, and let there be a tetrahedron $BDAA$ in the pseudotriangulation, with its edge AA winding one time around the knot (cf. the remark before this theorem). The move $1 \to 2$ is defined the following way: take a point C in the edge BD and replace the tetrahedron $BDAA$ by two tetrahedra $BCAA$ and $CDAA$. The move $2 \to 1$ is the inverse to that.*

Proof of this theorem is not very difficult for the careful reader of paper [2]. It will be presented in a further publication.

3   Acyclic complex

To a representation $f\colon \pi(K) \to G$ of the knot K group into some group G, a branched over K covering of sphere S^3 (where K lies) naturally corresponds. Namely, take first the *universal* branched over K covering of S^3 (where $\pi(K)$ naturally acts), and then identify its points x and x' if and only if x' is obtained from x by the action of an element of the subgroup $\operatorname{Ker} f \subset \pi(K)$. We will assume that f is *nontrivial*: $\operatorname{Im} f \neq \{e\}$, where e is the unit element of $\pi(K)$. Note that for f to be nontrivial it is sufficient that the image of some *overpass* of some knot K diagram, considered as an element of $\pi(K)$, should not equal e (then this will also hold for any overpass and any diagram).

Below we confine ourselves to the case $G = \mathrm{SO}(3)$. Then, the image of any overpass is a rotation through (one and the same) angle φ about some axis in the three-dimensional Euclidean space $\mathbb{R}^3$, all these axes going through the origin of coordinates. If there is only one such axis, then f is a *scalar* representation, i.e., a representation of the abelianization of $\pi(K)$.

We lift up the triangulation (or pseudotriangulation, we omit the prefix 'pseudo-' below) of S^3 satisfying the properties (a)–(c) from section 2 into the covering corresponding to representation f. Then we follow the same way as when constructing manifold invariants in papers [4, 7, 8, 9]: to each vertex of the lifted triangulation we ascribe coordinates in $\mathbb{R}^3$ obeying the following condition: if vertices $F^{(1)}$ and $F^{(2)}$ lie above the same vertex F of the initial triangulation of S^3 and $F^{(2)} = gF^{(1)}$, where $g \in \pi(K)$, then the coordinates of $F^{(2)}$ are obtained from those of $F^{(1)}$ by the transformation $f(g) \in \mathrm{SO}(3)$. One more condition is that of a "general position": any four vertices, if at least two of them do not belong to the knot, must not lie in the same (2-dimensional) plane (thus, the volumes of tetrahedra considered below will be nonzero). Otherwise, the coordinates we ascribe to vertices are arbitrary.

In such way, every edge of the *initial* triangulation acquires a Euclidean length, and usual Euclidean values are also assigned to dihedral angles and tetrahedron volumes. Some of these angles and volumes are taken, however, with the minus sign according to the following rule.

We assume that all the tetrahedra of our triangulation of S^3 are *oriented consistently*, i.e., for every tetrahedron, an order of its vertices is fixed up to even permu-

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tations; if there are two tetrahedra with a common two-dimensional face ABC and respective fourth vertices D and E, then their consistent orientations will be, for instance, $ABCD$ and $EABC$. When we place in $\mathbb{R}^3$ an oriented tetrahedron $ABCD$, we take as its volume the *oriented volume* $V_{ABCD} = \frac{1}{6}\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD}$ (triple scalar product). Recall that $V_{ABCD} \neq 0$. So, the sign of the volume is already here, and we assign to *all the dihedral angles of a tetrahedron* the same sign as its volume has.

Here is our acyclic complex (explanations below):

$$0 \to \mathfrak{a} \xrightarrow{\left(\frac{\partial x_i}{\partial(z \text{ or } \varphi)}\right)} (dx) \xrightarrow{\left(\frac{\partial l_a}{\partial x_i}\right)} (dl) \xrightarrow{\left(\frac{\partial \omega_b}{\partial l_a}\right)} (d\omega) \to (\cdots) \to (\cdots) \to 0. \qquad (1)$$

It resembles very much the complex used in our construction of three-manifold invariants in [9, formula (1)]. Like in that paper, we are using somewhat loose but very convenient notations for vector spaces consisting of differentials of Euclidean values.

Let us begin explanations from the space denoted (dx). It consists of column vectors whose entries are: three differentials dx_i, dy_i, dz_i of the coordinates of every vertex i which does not lie on the knot, and one differential dz'_k for every vertex k that does lie on the knot. To be exact, we choose *one* preimage in our lift-up for each vertex in the triangulation of S^3, and denote its coordinates x_i, y_i and z_i if this i-th vertex does not lie on the knot, or take just one coordinate z'_k if this k-th vertex does lie on the knot and hence its preimage lies on some fixed axis z' in $\mathbb{R}^3$ (z' is of course one of the axes mentioned in the second paragraph of this section).

The space (dl) consists of column vectors $(dl_1, \ldots, dl_{N_1})^{\mathrm{T}}$, where l_a is the length of the a-th edge and N_1 is the number of edges. It is clear how we define the mapping $(dx) \to (dl)$: it is the *differential* of the mapping that sends the coordinates of vertices to lengths of edges which join them.

Similarly, $(dl) \to (d\omega)$ is the differential of the mapping sending the edge lengths to the deficit angles around the edges. By definition, ω_b is minus algebraic sum of dihedral angles in all the tetrahedra which contain the edge b (it is here that the angle signs come into play). Here, the lengths l_a are allowed to have arbitrary infinitesimal increments dl_a, so the values ω_b deviate by some $d\omega_b$ from either $0 \pmod{2\pi}$ or $(-\varphi) \pmod{2\pi}$ for the respective edges lying or not lying on the knot.

Now we turn to the vector space $\mathfrak{a}$ and mapping $\mathfrak{a} \to (dx)$. By definition, $\mathfrak{a} = \{0\}$ if f is not a scalar representation. In the case f is scalar, we assume that f sends $\pi(K)$ to rotations about the axis z. Then $\mathfrak{a}$ is, by definition, the two-dimensional vector space consisting of columns denoted $\begin{pmatrix} dz \\ d\varphi \end{pmatrix}$. The mapping $\mathfrak{a} \to (dx)$, by definition, adds dz to the z-coordinates of all vertices and rotates them all through the angle $d\varphi$ around axis z.

In all these spaces $\mathfrak{a}$, (dx), (dl) and $(d\omega)$ there are natural distinguished bases (up to the ordering of basis vectors). Thus, all the mappings considered above are identified with their matrices. Moreover, matrix $\left(\frac{\partial \omega_b}{\partial l_a}\right)$ is *symmetric* [3, section 2], so the remaining arrows in the sequence (1) are filled with matrices transposed to $\left(\frac{\partial l_a}{\partial x_i}\right)$

and $\left(\frac{\partial x_i}{\partial(z \text{ or } \varphi)}\right)$, with no notice to the geometric meaning of the spaces denoted $(\cdots)$, in the same way as it has been done in [8, 9].

Theorem 2 *The sequence (1) is an acyclic complex.*

Proof. The fact that the product of two consecutive arrows is zero, follows immediately from geometric considerations. For instance, here is how they go for the two arrows at the term (dx): if lengths l_a are calculated from the vertex coordinates x_i (and do not change arbitrarily by themselves), then the deficit angles remain zero for edges not lying on the knot and equal to $(-\varphi)$ — for those lying there. On passing to infinitesimals, we get at once that the product of these two arrows is zero.

Thus we conclude that (1) is indeed an algebraic complex. As for the acyclicity, it can be proved using methods of paper [8], see section 2 of that work. The detailed proof will be presented in a further publication.

Our knot invariant is defined by the formula

$$I(K) = \tau \, \frac{\prod' l^2}{\prod 6V} \left(-2(1 - \cos \varphi)\right)^{N_0^{\text{knot}}}. \tag{2}$$

Here τ is the torsion of the complex (1). In terms of some minors of matrices (chosen according to the general theory of torsions), and taking into account the symmetry of the complex, it can be written as

$$\tau = \frac{(\text{minor}\left(\frac{\partial l_a}{\partial x_i}\right))^2}{\text{minor}\left(\frac{\partial \omega_b}{\partial l_a}\right)(\text{minor}\left(\frac{\partial x_i}{\partial z \text{ or } \varphi}\right))^2}; \tag{3}$$

the empty minor is considered to equal unity. The product in the denominator in (2) is taken over all tetrahedra, while the primed product in the numerator — only over the edges *not lying on the knot*. The number N_0^{knot} means the number of vertices in the triangulation *lying on the knot*.

Theorem 3 *The quantity I(K) is a knot invariant.*

Proof is based on investigating how the torsion behaves under the simplicial moves from Theorem 1. For moves $2 \leftrightarrow 3$ and $1 \leftrightarrow 4$ this has been already done, essentially, in [3], see also the parts of [8, 9] dealing with three-manifolds. As for moves $1 \leftrightarrow 2$, we are concerned with them in the next section.

4 Algebraic relation for move $1 \to 2$

Let a tetrahedron $BDAA$ be as in Theorem 1: edge BD lies on the knot K, while edge AA goes one time around K. Let us have a representation $f: \pi(K) \to \text{SO}(3)$

such that the passing along AA is taken into the rotation about the axis z through an angle $\varphi \neq 0 \pmod{2\pi}$. When lifted to the covering corresponding to f (see the first paragraph of section 3), the beginning and end of edge AA become different points, denoted below as $A^{(1)}$ and $A^{(2)}$.

Our aim now is to investigate how the torsion of complex (1) behaves under the move $1 \to 2$ described in Theorem 1. As the form of our acyclic complex (1) suggests, we should be interested in vertices, edges and tetrahedra taken away or added to the simplicial complex. Namely, the move $1 \to 2$:

- adds new vertex C;

- takes away the edge BD and adds new edges BC, CD and CA;

- takes away tetrahedron $BDAA$ and adds new ones, $BCAA$ and $CDAA$.

Considerations related to the triangular form of matrices and similar to those used in [3] show that the factor by which the torsion of complex (1) is multiplied under the move $1 \to 2$ can be calculated by using the following "local" acyclic complexes:

$$0 \to 0 \to (dl_{BD}) \to (d\omega_{BD}) \to 0 \to 0 \tag{4}$$

and

$$0 \to (dz_C) \to (dl_{BC}, dl_{CD}, dl_{AC}) \to (d\omega_{BC}, d\omega_{CD}, d\omega_{AC})$$
$$\to (\text{the term symmetric to } dz_C) \to 0. \tag{5}$$

Here, for instance, the term (dz_C) means the one-dimensional space — just the differential of coordinate z_C; $(dl_{BC}, dl_{CD}, dl_{AC})$ is a three-dimensional space, etc.

We denote the torsion of complex (4) as τ_1, and the torsion of complex (5) — as τ_2. Direct calculation shows that

$$\frac{\tau_2}{\tau_1} = \frac{6V_{CAAB} \cdot 6V_{DAAC}}{2(1 - \cos\varphi) \cdot l_{AC}^2 \cdot 6V_{DAAB}} \tag{6}$$

(and (6) also gives the ratio of torsions of the complexes corresponding to the whole triangulations of S^3 differing in the $1 \to 2$ move). The meaning of the result (6) is the following: the torsion (3) of complex (1):

- acquires the factor $-\frac{1}{2(1-\cos\varphi)}$ on adding a vertex (C in our case) lying on the knot;

- acquires the factor l_i^{-2} on adding an edge i not lying on the knot. Note that for edges *lying* on the knot, there are no such factors;

- acquires the factor $6V$ for every appearing tetrahedron (of volume V), and the factor $(6V)^{-1}$ for every disappearing tetrahedron.

This all together guarantees the invariance of value $I(K)$ under moves $1 \leftrightarrow 2$, which concludes the proof of Theorem 3.

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5 Discussion of results

The main thing in this paper is formula (6). It shows that a relation corresponding to a relevant simplicial move does exist for a nonzero deficit angle as well. In the present paper, this fact is applied to knot theory. On the other hand, it may find applications also in Regge calculus, i.e., discrete gravity theory.

It may make sense to search for a quantum analogue of formula (6). This problem seems to be solvable in our case of invariants built using Euclidean geometry in 3-simplices, because of analogy with the relation for move $2 \to 3$, which played the key role in constructing *manifold* invariants, started in [3], and which can be obtained in a (double) semiclassical limit from the pentagon equation for quantum $6j$-symbols.

One more apparently solvable problem is generalization of our geometric constructions onto the case of multidimensional analogues of knots, and onto the case where we apply some other than Euclidean geometry. Much more difficult looks the problem of *quantizing* of these latter constructions.

The results of actual calculations of our invariants $I(K)$ for specific knots K and representations of group $\pi(K)$ will be presented in E.V. Martyushev's work [1]. Also, in a future paper of larger size, complete proofs of our theorems will be given. Here we just mention that

$$I(\text{unknot}) = -4(1 - \cos \varphi)^2,$$

(which is not hard to calculate even without a computer) and that calculations show a connection with Alexander polynomial and Reidemeister torsion (both abelian and nonabelian). Most interesting results are expected for most complicated representations of $\pi(K)$ in some group of geometric origin, as the analogy with our manifold invariants [5, 6] suggests.

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